MEAN DAN VARIANSI DARI DISTRIBUSI NORMAL TERPOTONG

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Info:

Jurnal MSA Vol. 3 No. 2
Edisi: Januari – Juni 2014
Artikel No.: 6
Halaman: 35 - 46
ISSN: 2355-083X
Prodi Matematika UINAM

ABSTRAK

Distribusi normal terpotong muncul akibat tidak dilakukan observasi pada bagian tertentu dari suatu populasi distribusi normal (*Normal Distribution*). Akibat pemotongan, secara otomatis karakteristik data seperti mean dan variansi juga ikut berubah. μ yang semula Mean berubah menjadi mean terpotong dan σ^2 yang semula variansi berubah menjadi variansi terpotong. Penelitian ini bertujuan untuk mengetahui bentuk persamaan ekspektasi *(mean)* dan variansi distribusi normal terpotong *(truncated normal distribution*.

Kata Kunci: Distribusi normal, distribusi normal standar, distribusi normal terpotong, ekspektasi

1. PENDAHULUAN

Suatu populasi yang telah diketahui berdistribusi normal akan berubah menjadi distribusi normal terpotong *(Truncated Normal Distribution)* jika terdapat pemotongan nilai tertentu. Data terpotong tersebut muncul akibat tidak dilakukan observasi pada bagian tertentu atau terdapat data hilang dari suatu populasi.

Sebagai contoh, penelitian tentang nilai penjualan yang digolongkan kedalam industri menengah di beberapa perusahaan. Data yang diperoleh dalam penelitian ini merupakan data terpotong, karena objek utama merupakan bagian tertentu dari populasi. Akibat pemotongan, secara otomatis data sampel yang digunakan berubah, dan karakteristik data seperti ekspektasi dan variansi juga ikut berubah. μ dan σ sebelum mengalami pemotongan biasa mean dan variansi dari distribusi normal, sedang mean dan variansi berturut-turut bernilai 0 dan 1 jika distribusi normal standar. Setelah mengalami pemotongan *(truncation),* mean dan variansi dari distribusi normal mengalami perubahan menjadi mean terpotong *(truncated mean)* dan variansi terpotong *(truncated varian)*.

Adanya pemotongan *(truncation)* menyebabkan ada 3 kemungkinan bentuk distribusi yang diperoleh, yaitu distribusi normal terpotong bawah *(left truncated normal distribution)*, distribusi normal terpotong atas *(right truncated normal distribution)*, dan distribusi normal terpotong atas-bawah *(doubly truncated normal distribution)*.

Tujuan dari penulisan ini adalah mengetahui bentuk persamaan ekspektasi X(*mean*) dan variansi dari jenis-jenis distribusi normal terpotong *(truncated normal distribution)*

1. DISTRIBUSI NORMAL TERPOTONG

Distribusi normal terpotong adalah distribusi normal dengan nilai peubah acak X terbatas pada interval *[a , b]* atau $a \le X \le b$. Titik a adalah titik terpotong di sebelah kiri (disebut juga titik terpotong kiri) dan titik b adalah titik terpotong kanan (disebut juga titik terpotong kanan).

Fungsi kepadatan peluang distribusi normal terpotong sebagai berikut.

$$f(x|a < x < b) = \frac{f(x)}{P(a < x < b)}$$

Dimana $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ adalah fungsi kepadatan peluang distribusi normal. Diberikan transformasi $z = \frac{x-\mu}{\sigma}$ sedemikian sehingga

$$f(x|a < x < b) = \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

Dimana $\phi(z)$ dan $\Phi(z)$ adalah fungsi kepadatan peluang dan fungsi distribusi kumulatif dari distribusi normal baku

2. DISTRIBUSI NORMAL TERPOTONG BAWAH

Definis 1:

Misalkan x adalah suatu peubah acak distribusi normal dengan fungsi kepadatan peluangnya sebagai berikut:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty \le x \le \infty$$

Jika nilai x di bawah suatu nilai a tidak dapat diobservasi, maka hasil distribusinya adalah distribusi normal terpotong bawah. Dengan *fkp* sebagai berikut:

$$f_{x/a}(x|x > a) = \begin{cases} \dfrac{f(x)}{\int_a^\infty f(x)\,dx}; & a \le x \le \infty \\ 0 \quad ; & -\infty \le x < a \end{cases}$$

Dimana *f(x)* adalah fungsi kepadatan peluang distribusi normal.

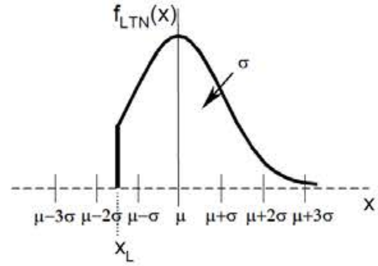
$$f(x|x > a) = \frac{f(x)}{P(a < x < \infty)}$$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{\infty-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi(\infty) - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$$f(x|x > a) = \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

Berikut kurva dari distribusi normal terpotong bawah:



Gambar 1 : Kurva distribusi normal terpotong bawah

Jika $X \sim N(\mu, \sigma^2)$ dan a konstanta, maka mean dan variansi dari distribusi normal terpotong bawah adalah sebagai berikut :

$$E\left(X \middle| X > a\right)$$

$$= \int_a^\infty x . f\left(x \middle| x > a\right) dx$$

$$= \lim_{b\to\infty} \int_a^b x . \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} dx$$

$$= \frac{1}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} \left[\lim_{b\to\infty} \int_{\frac{a-\mu}{\sigma}}^b \left(\sigma z + \mu\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \right]$$

$$= \frac{1}{1 - \Phi\left(\frac{a-\mu}{\sigma}\right)} \left[\left(\mu\left(1 - \Phi\left(\frac{a-\mu}{\sigma}\right)\right)\right) + \left(\frac{\sigma}{\sqrt{2\pi}}\left(\lim_{b\to\infty} \int_{\frac{a-\mu}{\sigma}}^b e^{-\frac{z^2}{2}} z dz\right)\right) \right]$$

$$= \frac{1}{1 - \Phi(\alpha)} \left[\left(\mu\left(1 - \Phi(\alpha)\right)\right) + \left(\frac{\sigma e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}\right) \right]$$

Jadi nilai meannnya adalah:

$$E(X|X > a) = \mu + \frac{\frac{\sigma e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}}{1-\Phi(\alpha)}$$

Nilai variansinya adalah :

$$E(X^2|X > a)$$

$$= \int_a^{\infty} x^2 . f(x|x > a)\, dx$$

$$= \lim_{b\to\infty} \int_a^b x^2 . \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}\, dx$$

$$= \frac{1}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}\left[\lim_{b\to\infty} \int_{\frac{a-\mu}{\sigma}}^{b} (\sigma z + \mu)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz\right]$$

$$= \frac{1}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}\left[\lim_{b\to\infty} \int_{\alpha}^{b} \left(\mu^2 + \sigma^2 z^2 + 2\mu\sigma z\right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz\right]$$

$$= \mu^2 + \frac{\sigma^2 \frac{\alpha e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)} + \sigma^2 + \frac{2\mu\sigma \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$$= \mu^2 + \frac{2\mu\sigma \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)} + \frac{\sigma^2 \frac{\alpha e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)} + \sigma^2$$

3.

$$Var(X|X > a)$$

$$= E\left(X^2\, X > a|\right) - \left[E\left(X|X > a\right)\right]^2$$

$$= \left[\mu^2 + \frac{2\mu\sigma M}{1-\Phi(\alpha)} + \frac{\sigma^2 \alpha M}{1-\Phi(\alpha)} + \sigma^2\right] + - \left[\mu + \frac{\sigma M}{1-\Phi(\alpha)}\right]^2$$

$$= \left[\mu^2 + \frac{2\mu\sigma M}{1-\Phi(\alpha)} + \frac{\sigma^2 \alpha M}{1-\Phi(\alpha)} + \sigma^2\right] + - \left[\mu^2 + \frac{2\mu\sigma M}{1-\Phi(\alpha)} + \left(1-\Phi(\alpha)\right)^2\right]$$

$$= \mu^2 + \frac{2\mu\sigma M}{1-\Phi(\alpha)} + \frac{\sigma^2 \alpha M}{1-\Phi(\alpha)} + \sigma^2 + - \mu^2 - \frac{2\mu\sigma M}{1-\Phi(\alpha)} - \left(\frac{\sigma M}{1-\Phi(\alpha)}\right)^2$$

$$= \frac{\sigma^2 \alpha M}{1-\Phi(\alpha)} + \sigma^2 - \left(\frac{\alpha M}{1-\Phi(\alpha)}\right)^2$$

$$= \frac{\sigma^2 \alpha M}{1-\Phi(\alpha)} + \sigma^2 - \sigma^2\left(\frac{M}{1-\Phi(\alpha)}\right)^2$$

$$= \sigma^2\left(\frac{\alpha M}{1-\Phi(\alpha)} + 1 - \left(\frac{M}{1-\Phi(\alpha)}\right)^2\right)$$

$$= \sigma^2\left(1 - \left(\frac{M}{1-\Phi(\alpha)}\right)^2 + \frac{\alpha M}{1-\Phi(\alpha)}\right)$$

$$= \sigma^2\left(1 - \frac{M}{1-\Phi(\alpha)}\left(\frac{M}{1-\Phi(\alpha)} - \alpha\right)\right)$$

Dengan,

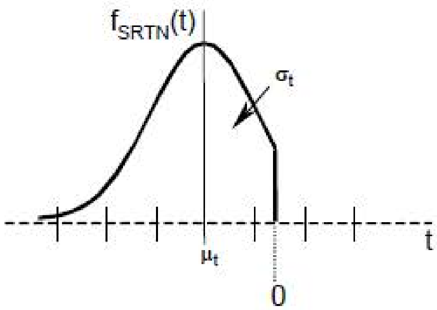
$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz$$

$$M = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}$$

$$\alpha = \frac{a-\mu}{\sigma}$$

3. DISTRIBUSI NORMAL TERPOTONG ATAS

Berikut kurva dari distribusi normal terpotong atas:



Gambar 2: Kurva distribusi normal terpotong atas.

Berbeda dengan distribusi normal terpotong bawah, jika nilai x yang berada di atas suatu nilai b tidak dapat diobservasi, maka hasil distribusinya adalah distribusi normal terpotong atas dengan fkp sebagai berikut :

$$f_{x/b}(x|x<b) = \begin{cases} \dfrac{f_1(x)}{\int_{-\infty}^{b} f_1(x)\,dx} & ; -\infty \le x \le b \\ 0 & ; \; b < x \le \infty \end{cases}$$

$$f\left(x|x<b\right) = \frac{f(x)}{P(-\infty < x < b)}$$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{-\infty-\mu}{\sigma}\right)}$$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right) - 1 - \Phi(\infty)}$$

$$= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right)}$$

Jika $X \sim N(\mu, \sigma^2)$ dan b konstanta, maka mean dan variansi dari distribusi normal terpotong atas adalah sebagai berikut :

$$E\left(X|X<b\right)$$

$$= \int_{-\infty}^{b} x \,.\, f\left(x|x<b\right) dx$$

$$= \lim_{a\to-\infty} \int_{a}^{b} x \,.\, \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right)} dx$$

$$= \frac{1}{\Phi\left(\frac{b-\mu}{\sigma}\right)} \left[\lim_{a\to-\infty} \int_{a}^{\frac{b-\mu}{\sigma}} (\sigma z + \mu) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \right]$$

$$= \frac{1}{\Phi(\beta)} \left[\lim_{a\to-\infty} \int_{a}^{\frac{b-\mu}{\sigma}} \sigma z \,.\, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz + \lim_{a\to-\infty} \int_{a}^{\frac{b-\mu}{\sigma}} \mu \,.\, \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \right]$$

$$= \frac{1}{\Phi\left(\frac{b-\mu}{\sigma}\right)} \left[\begin{array}{l} \left(\mu\left(\Phi\left(\frac{b-\mu}{\sigma}\right)\right)\right) + \\ -\left(\sigma \,.\, \lim_{a\to-\infty} \int_{a}^{\frac{b-\mu}{\sigma}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} d\left(-\frac{z^2}{2}\right)\right) \end{array} \right]$$

$$= \frac{1}{\Phi\left(\frac{b-\mu}{\sigma}\right)} \left[\mu\left(\Phi\left(\frac{b-\mu}{\sigma}\right)\right) - \frac{\sigma e^{-\frac{\left(\frac{b-\mu}{\sigma}\right)^2}{2}}}{\sqrt{2\pi}} \right]$$

$$= \mu - \frac{\frac{\sigma e^{-\frac{\left(\frac{b-\mu}{\sigma}\right)^2}{2}}}{\sqrt{2\pi}}}{\Phi\left(\frac{b-\mu}{\sigma}\right)}$$

$$E\left(X^2 \middle| X < b\right)$$

$$= \int_{-\infty}^{b} x^2 . f\left(x \middle| x < b\right) dx$$

$$= \lim_{a \to -\infty} \int_a^b x^2 . \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right)} dx$$

$$= \frac{1}{\Phi\left(\frac{b-\mu}{\sigma}\right)} \left[\lim_{a \to -\infty} \int_a^b x^2 . \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \right]$$

$$= \frac{1}{\Phi\left(\frac{b-\mu}{\sigma}\right)} \left[\lim_{a \to -\infty} \int_a^{\frac{b-\mu}{\sigma}} (\sigma z + \mu)^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \right]$$

$$= \frac{1}{\Phi(\beta)} \left[\begin{array}{l} \mu^2 \Phi(\beta) + \sigma^2 \lim_{a \to -\infty} \int_a^{\beta} z^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz + \\ -2\mu\sigma \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}} \end{array} \right]$$

$$= \frac{1}{\Phi(\beta)} \left[\begin{array}{l} \mu^2 \Phi(\beta) + \sigma^2 \left(\frac{-\beta e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}} + \Phi(\beta) \right) + \\ -\frac{2\mu\sigma e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}} \end{array} \right]$$

$$= \mu^2 - \frac{\frac{2\mu\sigma e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}}}{\Phi(\beta)} - \frac{\frac{\sigma^2 \beta e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}}}{\Phi(\beta)} + \sigma^2$$

$$Var\left(X \middle| X < b\right)$$

$$= E\left(X^2 \middle| X < b\right) - \left[E\left(X \middle| X < b\right)\right]^2$$

$$= \left[\mu^2 - \frac{2\mu\sigma N}{\Phi(\beta)} - \frac{\sigma^2 \beta N}{\Phi(\beta)} + \sigma^2\right] - \left[\mu - \frac{\sigma N}{\Phi(\beta)}\right]^2$$

$$= \mu^2 - \frac{2\mu\sigma N}{\Phi(\beta)} - \frac{\sigma^2 \beta N}{\Phi(\beta)} + \sigma^2 +$$

$$- \left[\mu^2 - \frac{2\mu\sigma N}{\Phi(\beta)} + \left(\frac{\sigma N}{\Phi(\beta)}\right)^2\right]$$

$$= \mu^2 - \frac{2\mu\sigma N}{\Phi(\beta)} - \frac{\sigma^2 \beta N}{\Phi(\beta)} + \sigma^2 - \mu^2 +$$

$$\frac{2\mu\sigma N}{\Phi(\beta)} - \left(\frac{\sigma N}{\Phi(\beta)}\right)^2$$

$$= -\frac{\sigma^2 \beta N}{\Phi(\beta)} + \sigma^2 - \left(\frac{\sigma N}{\Phi(\beta)}\right)^2$$

$$= \sigma^2 - \frac{\sigma^2 \beta N}{\Phi(\beta)} - \sigma^2 \left(\frac{N}{\Phi(\beta)}\right)^2$$

$$= \sigma^2 \left[1 - \frac{\beta N}{\Phi(\beta)} - \left(\frac{N}{\Phi(\beta)}\right)^2\right]$$

$$= \sigma^2 \left[1 - \frac{N}{\Phi(\beta)}\left(\beta + \frac{N}{\Phi(\beta)}\right)\right]$$

Dengan

$$\Phi(\beta) = \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$N = \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}}$$

$$\beta = \frac{b-\mu}{\sigma}$$

4. DISTRIBUSI NORMAL TERPOTONG ATAS BAWAH

Berdasar variabel acak berdistribusi normal dengan fkp sebagai berikut:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, -\infty \le x \le \infty$$

jika nilai x di bawah suatu nilai a dan di atas suatu nilai b tidak dapat diobservasi, maka hasil distribusi yang diperoleh adalah distribusi normal terpotong atas-bawah dengan fkp sebagai berikut:

$$f_{x/b}(x|x<b)=f(x)=\begin{cases}0 \quad , & -\infty<x\le a\\ \dfrac{f_1(x)}{\int_a^b f_1(x)dx}, & a<x\le b\\ 0 \quad , & b<x\le\infty\end{cases}$$

Dimana *f(x)* adalah fungsi kepadatan peluang distribusi normal.

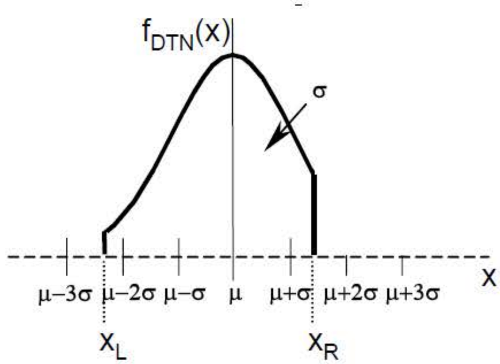
$$f(x|a<x<b)=\frac{f(x)}{P(a<x<b)}$$

$$=\frac{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$$=\frac{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$$=\frac{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)}$$

Berikut kurva dari distribusi normal terpotong bawah:



Gambar 3: Kurva distribusi normal terpotong atas-bawah

Jika $X\sim N(\mu,\sigma^2)$ dan a, b konstanta, maka mean dan variansi dari distribusi normal terpotong atas bawah adalah sebagai berikut :

$$E(X|a<X<b)$$
$$=\int_a^b x\,.\,f(x|a<x<b)\,dx$$
$$=\int_a^b x\,.\,\frac{\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi(\beta)-\Phi(\alpha)}dx$$
$$=\frac{1}{\Phi(\beta)-\Phi(\alpha)}\left[\int_\alpha^\beta(\sigma z+\mu)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right]$$
$$=\frac{1}{\Phi(\beta)-\Phi(\alpha)}[(\Phi(\beta)-\sigma N)-(\Phi(\alpha)-\sigma M)]$$
$$=\frac{1}{\Phi(\beta)-\Phi(\alpha)}[\mu(\Phi(\beta)-\Phi(\alpha))-\sigma(N-M)]$$
$$=\frac{\mu(\Phi(\beta)-\Phi(\alpha))}{\Phi(\beta)-\Phi(\alpha)}-\frac{\sigma(N-M)}{\Phi(\beta)-\Phi(\alpha)}$$
$$=\mu-\frac{\sigma(N-M)}{\Phi(\beta)-\Phi(\alpha)}$$

Dengan,

$$\alpha=\frac{a-\mu}{\sigma}$$

$$\beta=\frac{b-\mu}{\sigma}$$

$$\Phi(\alpha)=\int_{-\infty}^{\alpha}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$\Phi(\beta)=\int_{-\infty}^{\beta}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$N=\frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}}$$

$$M=\frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}$$

$$E\left(X^2 \middle| a < X < b\right) = \int_a^b x^2 . f\left(x \middle| x < b\right) dx$$

$$= \int_a^b x^2 . \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\Phi(\beta) - \Phi(\alpha)} dx$$

$$= \frac{1}{\Phi(\beta) - \Phi(\alpha)} \left[\int_\alpha^\beta \left(\mu^2 + \sigma^2 z^2 + 2\mu\sigma z\right) \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \right]$$

$$= \mu^2 - \frac{\sigma^2(\beta N - \alpha M)}{\Phi(\beta) - \Phi(\alpha)} + \sigma^2 - \frac{2\mu\sigma(N - M)}{\Phi(\beta) - \Phi(\alpha)}$$

$$= \mu^2 + \sigma^2 - \frac{2\mu\sigma(N - M)}{\Phi(\beta) - \Phi(\alpha)} - \frac{\sigma^2(\beta N - \alpha M)}{\Phi(\beta) - \Phi(\alpha)}$$

$$Var\left(X \middle| a < X < b\right)$$

$$= E\left(X^2 \middle| a < X < b\right) - \left[E\left(X \middle| a < X < b\right)\right]^2$$

$$= \mu^2 + \sigma^2 - \frac{2\mu\sigma(N - M)}{\Phi(\beta) - \Phi(\alpha)} - \frac{\sigma^2(\beta N - \alpha M)}{\Phi(\beta) - \Phi(\alpha)} + -\left(\mu - \frac{\sigma(N - M)}{\Phi(\beta) - \Phi(\alpha)}\right)^2$$

$$= \sigma^2 - \frac{\sigma^2(\beta N - \alpha M)}{\Phi(\beta) - \Phi(\alpha)} - \left(\frac{\sigma(N - M)}{\Phi(\beta) - \Phi(\alpha)}\right)^2$$

$$= \sigma^2 \left[1 + \frac{2NM}{\left(\Phi(\beta) - \Phi(\alpha)\right)^2}\right] - \sigma^2 \mathrm{x} \left[\left[\frac{N}{\Phi(\beta) - \Phi(\alpha)}\left(\frac{N}{\Phi(\beta) - \Phi(\alpha)} + \beta\right)\right] + \left[\frac{M}{\Phi(\beta) - \Phi(\alpha)}\left(\frac{M}{\Phi(\beta) - \Phi(\alpha)} - \alpha\right)\right]\right]$$

Dengan,

$$\alpha = \frac{a - \mu}{\sigma}$$

$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$M = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}$$

$$\beta = \frac{b - \mu}{\sigma} \quad \Phi(\beta) = \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \quad N = \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}}$$

5. PENUTUP

Mean dan variansi terpotong dari distribusi normal terpotong bawah adalah $E\left(X \middle| X > a\right) = \mu + \frac{\sigma M}{1 - \Phi(\alpha)}$ dan $Var\left(X \middle| X > a\right) = \sigma^2\left[1 - \frac{M}{1 - \Phi(\alpha)}\left(\frac{M}{1 - \Phi(\alpha)} - \alpha\right)\right]$,

Mean dan variansi terpotong dari distribusi normal terpotong atas adalah $E\left(X \middle| X < b\right) = \mu - \frac{\sigma N}{\Phi(\beta)}$ dan $Var\left(X \middle| X < b\right) = \sigma^2\left[1 - \frac{N}{\Phi(\beta)}\left(\frac{N}{\Phi(\beta)} + \beta\right)\right]$,

Mean dan variansi terpotong dari distribusi normal terpotong atas-bawah adalah $E\left(X \middle| a < X < b\right) = \mu - \frac{\sigma(N - M)}{\Phi(\beta) - \Phi(\alpha)}$ dan

$$Var\left(X \middle| a < X < b\right) = \sigma^2\left[1 + \frac{2NM}{\left(\Phi(\beta) - \Phi(\alpha)\right)^2}\right] - \sigma^2 \left[\begin{matrix}\left[\frac{N}{\Phi(\beta) - \Phi(\alpha)}\left(\frac{N}{\Phi(\beta) - \Phi(\alpha)} + \beta\right)\right] + \\ \left[\frac{M}{\Phi(\beta) - \Phi(\alpha)}\left(\frac{M}{\Phi(\beta) - \Phi(\alpha)} - \alpha\right)\right]\end{matrix}\right],$$

dengan $\alpha = \frac{a - \mu}{\sigma}$, $\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$, $M = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}}$, $\beta = \frac{b - \mu}{\sigma}$, $\Phi(\beta) = \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$ dan $N = \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}}$.

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